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Math Expressions

Next-Generation NSF Program

Math Talk Math Talk Student Concepts Building Quick Practice

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Math Expressions

NEXT-GENERATION NSF PROGRAM

A Research-Based Framework for *Math Expressions*

Grades K–5

HOUGHTON MIFFLIN

Introduction

Adding It Up: Helping Children Learn Mathematics, a report published by the National Research Council in 2001, is a summary of the work of a committee of members from diverse backgrounds who reviewed and synthesized relevant research on mathematics learning from Pre-Kindergarten through Grade 8 [National Research Council (NRC), 2001, p. ix].

The committee chose the term "mathematical proficiency" to capture what it means for anyone to learn mathematics successfully. Mathematical proficiency is broken down into the following five components, or strands:

- **Conceptual understanding**—Comprehension of mathematical concepts, operations, and relations
- **Procedural fluency**—Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic competence**—Ability to formulate, represent, and solve mathematical problems
- **Adaptive reasoning**—Capacity for logical thought, reflection, explanation, and justification
- **Productive disposition**—Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

In the short summary of the longer report, these strands are termed **understanding, computing, applying, reasoning**, and **engaging** (NRC, 2002).

The most important observation the committee makes and stresses is that the **five strands are interwoven and interdependent in the development of proficiency in mathematics** [NRC, 2001, p. 116]. The information on the research base for ***Math Expressions*** is organized around these five strands.

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Background

Math Expressions is based on the research results of the *Children's Math Worlds* (*CMW*) research project. Both the program and the research have a focus on conceptual understanding intertwined with the other components of math proficiency. Dr. Karen C. Fuson, now professor emeritus of learning sciences at Northwestern University, Evanston, Illinois, is the principal investigator of *CMW* and author of ***Math Expressions***.

In this complete K–5 curriculum, Grades K and 1 consist of completely integrated units. Grades 2–5 employ longer units that focus on and relate word problems, computation, algebra, and data. Each unit in Grades 2–5 is followed by a mini-unit that focuses on geometry or measurement and uses previously learned mathematics. This organization is a result of teacher and student feedback during the *CMW* research project. At each grade level, the program includes a two-volume Teacher's Guide, two consumable Student Activity Books, two consumable Homework and Remembering Workbooks, a Teacher's Resource Book, an Assessment Guide, MathBoards, and various specially designed manipulatives and other learning materials.

CMW classroom research with teachers identified five crucial components of a classroom that develop mathematical understanding, competence, and confidence. The components—Building Concepts, Math Talk, Student Leaders, Quick Practice, and Building Community—interact synergistically in the classroom. Implementing these components enables children from all backgrounds to learn mathematics.

THE *MATH EXPRESSIONS* PHILOSOPHY

Math Expressions was developed to meet the national need for a balanced program that can:

- Combine a reform math program focus on understanding with a more traditional program focus on skill
- Use an approach that emphasizes in-depth, sustained learning of core grade-level concepts (rather than a spiral curriculum that does not support conceptual understanding and fluency)
- Expand the types of word problems to include those solved by students in other countries
- Use math drawings made by students and research-based visual representations in each math domain to support student understanding and class discussion of mathematical thinking
- Find and use computational algorithms that relate easily to common U.S. algorithms but that can also be understood and used more easily by students
- Start at the students' level and continually elicit their thinking, but provide visual and linguistic supports that all children understand
- Bring all children to fluency in the core elementary computational topics while still incorporating important parts of other mathematical topics

Among the key research factors influencing the program are: the power of student drawings to express math thinking and support math discussions, the kinds of algorithms that fit student thinking and relate easily to current methods, developmental sequences of student strategies in math domains, student conceptual language, visual representations to support understanding, and types of word problems. Features of the program that reflect this research include coherent in-depth curricular learning paths, ongoing interactions between individual and whole-class learning, differentiated instruction within whole-class activities, and research-based models, strategies, and algorithms that help all students.

Research Base

Math Expressions is the only U.S. curriculum developed using the methods of learning science design research. Work was done in classrooms for at least four to five years on major topics at each grade level, with continual revision of the teaching/learning materials. The goal was to identify supported learning paths through major math domains that could be coherently woven across grades. The research tasks included:

- Analyzing real-world mathematical situations to help curriculum developers and teachers select problems and examples that ensure both the understanding of the general math principles at work and of the real-world situation itself
- Analyzing formal mathematical language and notation to identify difficulties that need to be addressed with pedagogical supports and classroom discussion
- Developing meaningful real-world situations and visual supports that can facilitate interest and accessibility
- Identifying meaningful language that can connect to the formal mathematical language (e.g., "break-apart partners" for addends, "unmultiplying" for dividing)
- Identifying typical student solution methods and learning paths through a domain to more-advanced solution methods
- Developing accessible algorithms that relate to common algorithms but that all students can understand and explain
- Identifying typical student errors and how to overcome them
- Choosing drawn quantity representations that can facilitate understanding of the domain situations or quantities
- Monitoring grade-level placements of, and approaches to, important topics around the world
- Writing teaching materials in a "learn while teaching" style that enables teachers to learn new ways of teaching and new solution methods
- Developing classroom activity structures that can be used repeatedly with different math topics to cut down on classroom management issues

One of the most important goals of developing ***Math Expressions*** is to facilitate the building of a teaching-learning community in the math classroom. In such a learning environment, students are made to feel safe, trusted, and validated—they feel like contributing community members. In such classrooms, competence and confidence will develop hand in hand. Teachers must help all students learn how to interact both autonomously and interdependently, creating a learning community in which all members are learners and teachers.

Overall, ***Math Expressions*** was designed to lead all students to mathematical proficiency by addressing all five strands determined by the National Research Council: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

CONCEPTUAL UNDERSTANDING

The *CMW* research spanned ten years, working with teachers, students, parents, and administrators to identify learning approaches accessible to all children. Built on the best of traditional and reform mathematics instruction, ***Math Expressions*** encourages connective teaching and math modeling (mathematical words and symbols linked to meaningful referents).

The research-based solution methods that are taught in the program help move students quickly to accurate and rapid-enough methods that are within the research-based learning path. Evidence indicates that understanding and skill do not develop separately but are continually intertwined. For example, it is more effective to intertwine student work on word and numerical problems than to do numerical problems separately first.

In ***Math Expressions,*** the teacher demonstrates and explains, but also encourages students to do the same, assisting them in learning productive roles in each classroom activity structure. Meaning-making activities build initial understanding, while additional practice encourages fluency. ***Math Expressions*** students use activity sheets and do homework, but initial work has visual learning supports that help students link their initial knowledge to the formal math. Students make math drawings initially to help them build understanding and support their explanations of their solution methods. Individual problem solving is encouraged, with help available from both peers and the teacher.

PROCEDURAL FLUENCY

Math Expressions constructs ladders for student learning, based on the best of established instruction and extended through new reform methods. Through practice, individual problem solving, worksheets, and homework, students develop mathematical fluency. As they develop this fluency, they are aided by the teacher, by their peers, by "home helpers," and by their continued use of meaningful math drawings and accessible algorithms to keep their understandings related to their increasingly fluent work.

Traditional programs teach the common algorithms—multi-step computational methods—that many adults may remember learning when they were in school. Some reform programs introduce alternative algorithms, some of which support understanding and fluency and some of which do not. ***Math Expressions*** instead uses accessible algorithms identified or developed during extensive classroom research. These new research-based algorithms relate readily to common algorithms, but they are more accessible to students. These methods help students build understanding of the central mathematical ideas involved in each kind of computation and notation. Students using ***Math Expressions*** make drawings of the math quantities and explain their methods to other students by relating their numerical steps to steps in their math drawings. ***Math Expressions*** always relates its own student-friendly algorithms to common methods, students' own methods, and methods from students' homes to facilitate discussion and understanding in the classroom.

Math Expressions does this for several reasons. First, our research has shown that students prefer to use different algorithms. Furthermore, having more than one algorithm allows students to discuss the concepts behind them, as well as their advantages and disadvantages. This approach allows all students to find, use, understand, and explain algorithms that work for them. Less-advanced students typically choose and use one algorithm. Other students may use and be able to explain more than one while becoming fluent with at least one. Students stop making math drawings whenever they can explain their numerical methods without drawings (they may still be asked to make math drawings when they explain their methods to other students). These visual supports, along with meaningful place-value language, enable everyone to participate in Math Talk. The use of mathematically general accessible algorithms enables all students to understand and become fluent.

STRATEGIC COMPETENCE

In ***Math Expressions,*** strategic competence is developed from the beginning by starting with real-world situations and their word-problem counterparts, and continually weaving these situations into other work.

During class, the teacher models the mathematizing—focusing on mathematical structure and relationships—that is the key to strategic competence. The teacher approaches math from students' points of view by eliciting their thinking and rich language use. The use of both common informal and formal mathematical language helps students develop the vocabulary needed to express their growing understanding, and also helps them link drawings to written math notation to facilitate math modeling. Reflection, discussion, and analysis during class are facilitated by the use of meaningful math drawings, which allow teachers to interpret students' thinking and errors. During this process, students continually develop and share their own language, questions, and problems with the class. In turn, problems will become accessible to all students through multiple levels of access. Such a classroom is called the referential classroom, because of the numerous visual and language referents for the formal mathematical ideas.

The process of using and validating students' own language and experiences while connecting them to standard language and symbols facilitates listening, speaking, writing, and helping competencies, in addition to improving math skill.

In ***Math Expressions,*** the use of visual representations to support conceptual understanding is pervasive. Initially, in many areas, special manipulatives are used. But, with the exception of Kindergarten, the program rapidly moves to the use of math drawings in different conceptual areas. Math drawings support Math Talk because they can be done on the board for everyone to see during explanations of math thinking. They leave a record of students' thinking during class work and homework so teachers can get a feel for student approaches and errors and what still needs to be discussed. They eliminate logistical and cost issues that arise with manipulatives and provide continual experience with two-dimensional spatial thinking. They allow students to take pride in drawings they produce and they help reduce attention-consuming issues (fiddling, dropping things, off-task work, etc.). Math drawings focus on core math ideas and structures, and they provide experiences with fundamental math notations and concepts such as length and area. They also make it easier to link the meanings in the drawing to numerical problems and computations. Such links can be made with circles, arrows, or other symbols.

In ***Math Expressions,*** word problems are continually intertwined with computational learning. As a result, students from all backgrounds are able to solve an ambitious and carefully chosen trajectory of word problem types across the grades. This algebraic approach to word problem solving involves two steps:

1. Use visual math models.

Read to comprehend the situation and make a math drawing (draw a math model) if it will help. Students also retell problems in their own words and explain their solutions, linking their drawings to elements in the problem.

Students first make their own math drawings by focusing on the mathematical aspects of the situation and by using circles or other simple shapes to show the situation.

They are then introduced to research-based accessible math models for each type of word-problem situation. These models also relate to or are math notations such as equations, areas, or tables.

Our visual models reflect the different meanings of = in different types of word-problem situations: = can mean "becomes" or "is identical to" or "is the same amount as." These different meanings of = traditionally have been the source of difficulties in algebra, so understanding them thoroughly is important.

2. Develop embedded numbers and learn subtraction as unknown addition and division as unknown multiplication.

Solving the more difficult word-problem types such as 8 + ? = 14 ("I had 8 mangoes and got some more. Now I have 14. How many mangoes did I get?") requires that students can conceptualize 14 as being made from 8 and 6. We begin in Kindergarten by developing understanding of such "break-apart partners" hiding inside a number. The first equation introduced in Kindergarten is of the form 5 = 3 + 2 (5 can be made from 3 and 2). Students explore all of the break-apart partners (addends) that make a given number. This is common in many other countries, but is often not emphasized in the United States.

Students find eight true equations for a given triad of numbers, rather than just the usual four in a "fact family." So for 7, 3, and 10, students generate and discuss the equations with only one number on the left (10 = 7+ 3, 10 = 3 + 7, 3 = 10 – 7, 7 = 10 – 3) as well as the usual four equations with one number on the right (7 + 3 = 10, 3 + 7 = 10, 10 – 7 = 3, 10 – 3 = 7).

ADAPTIVE REASONING

In ***Math Expressions,*** competency and fluency in mathematics do not come through problem solving and written practice alone. One of the most important aspects of this program is developing a collaborative Math Talk culture of understanding, explaining, questioning, and helping.

During this program's research stage, students' effective language and solution methods (e.g., division is "unmultiplying") were gathered and woven into the learning materials. We found that continual focus on sense making and on helping and explaining within a classroom community facilitated language development, competence, and confidence.

In the ***Math Expressions*** classroom, learning occurs within a supportive cognitive and emotional environment in which help is available both from the teacher and from peers. This is called connected knowing. Similarly, opportunities for separate knowing are encouraged, so that students can modify or improve others' ideas in a respectful, sensitive, and helpful way. The "de-bugging" of errors helps everyone—student and teacher—learn more deeply.

Math Talk connects to math drawings and story situations to help students learn math both visually and orally. As students come to understand mathematical actions and the use of meaningful math drawings, they also practice and become proficient in verbalizing these understandings. Students often explain their thinking by describing aspects of their own math drawings. They analyze each other's different methods of solving problems and help each other understand by explaining their way of thinking about a problem or procedure. In this classroom, all teachers are learners and all students are learners, as well as teachers of themselves and their peers.

Math Expressions introduces a Solve, Explain, Question, and Justify classroom structure in which students make math drawings at the board along with their numerical solution method. Then, two or three students explain their methods while other students ask questions to stimulate more complete and adapted explanations. The teacher facilitates from the side or the back of the room to increase the amount of direct student-to-student dialogue. Initially, for any new math topic, the teacher may also need to model full explanations of some methods and help students explain more fully. Each Teacher's Guide also contains sample questions, explanations, and student-teacher dialogue to help teachers build a more advanced Math Talk classroom.

<table>
<tr><th colspan="4">Levels of Math Talk Learning Community: Teacher and Student Action Trajectories</th></tr>
<tr><th colspan="4">Components of the Math Talk Learning Community</th></tr>
<tr><th>A. Questioning</th><th>B. Explaining Math Thinking</th><th>C. Source of Math Ideas</th><th>D. Responsibility for Learning</th></tr>
<tr><td colspan="4">Overview of shift among Levels 0–3: The classroom community grows to support students' acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking.</td></tr>
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<tr><td colspan="4">Level 0: Traditional teacher-directed classroom with brief answer responses from students.</td></tr>
<tr><td>*Teacher is the only questioner. Short, frequent questions function to keep students listening and paying attention to the teacher.*
Students give short answers and respond to the teacher only. No student-to-student Math Talk.</td><td>*Minimal or no teacher elicitation of student thinking, strategies, or explanations; teacher expects answer-focused responses. Teacher may tell answers.*
No student thinking or strategy-focused explanation of work. Only answers are given.</td><td>*Teacher is physically at the board, usually chalk in hand, telling and showing students how to do math.*
Students respond to math presented by the teacher. They do not offer their own math ideas.</td><td>*Teacher repeats student responses (originally directed to him/her) for the class. Teacher responds to students' answers by verifying the correct answer or showing the correct method.*
Students are passive listeners; they attempt to imitate the teacher and do not take responsibility for the learning of their peers or themselves.</td></tr>
<tr><td colspan="4">Level 1: Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the Math Talk community.</td></tr>
<tr><td>*Teacher questions begin to focus on student thinking and focus less on answers. Teacher begins to ask follow-up questions about student methods and answers. Teacher is still the only questioner.*
As a student answers a question, other students listen passively or wait for their turn.</td><td>*Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in explanations herself.*
Students give information about their math thinking, usually probed by the teacher (minimal volunteering of thoughts). They provide brief descriptions of their thinking.</td><td>*Teacher is still the main source of ideas, though he/she elicits some student ideas. Teacher does some probing to access student ideas.*
Some student ideas are raised in discussions, but are not explored.</td><td>*Teacher begins to set up structures to facilitate students listening to and helping other students. The teacher alone gives feedback.*
Students become more engaged by repeating what other students say or by helping another student at the teacher's request. This helping mostly involves students showing how they solved a problem.</td></tr>
<tr><td colspan="4">Level 2: Teacher modeling and helping students build new roles. Some coteaching and colearning begins as student-to-student talk increases. Teacher physically begins to move to side or back of room.</td></tr>
<tr><td>*Teacher continues to ask probing questions and also asks more open questions. He/she also facilitates student-to-student talk by asking students to be prepared to ask questions about other students' work.*
Students ask questions of one another's work on the board, often at the prompting of the teacher. Students listen to one another so they do not repeat questions.</td><td>*Teacher probes more deeply to learn about student thinking and supports detailed descriptions from students. Teacher welcomes and elicits multiple strategies.*
Students usually give information, mostly probed by the teacher with some volunteering of thoughts. They begin to stake a position and articulate more information in response to probes. They explain steps in their thinking by providing fuller descriptions and begin to defend their answers and methods. Other students listen supportively.</td><td>*Teacher follows up on explanations and builds on them by asking students to compare and contrast them. Teacher is comfortable using student errors as opportunities for learning.*
Students exhibit confidence about their ideas and share their own thinking and strategies even if they are different from others. Student ideas sometimes guide the direction of the math lesson.</td><td>*Teacher encourages student responsibility for understanding the mathematical ideas of others. Teacher asks other students questions about student work and whether they agree or disagree and why.*
Students listen and begin to understand one another. When the teacher requests, they explain other students' ideas in their own words. Helping involves clarifying other students' ideas for themselves and others. Students imitate and model teacher's probing in pair work and in whole-class discussions.</td></tr>
<tr><td colspan="4">Level 3: Teacher as coteacher and colearner. Teacher monitors all that occurs, still fully engaged. Teacher is ready to assist, but now in more peripheral and monitoring role (coach and assister).</td></tr>
<tr><td>*Teacher expects students to ask one another questions about their work. The teacher's questions still may guide the discourse.*
Student-to-student talk is student-initiated, not dependent on the teacher. Students ask questions and listen to responses. Many questions are "why" questions that require justification from the person answering. Students repeat their own or others' questions until satisfied with answers.</td><td>*Teacher follows closely students' descriptions of their thinking; may ask probing questions to make explanations more complete. Teacher stimulates students to think more deeply about strategies.*
Students describe more complete strategies; they defend and justify their answers with little prompting from the teacher. Students realize that they will be asked questions from other students when they finish, so they are motivated and careful to be thorough. Other students support with active listening.</td><td>*Teacher allows for interruptions from students during explanations; lets students explain and "own" new strategies. (Teacher is still engaged and deciding what is important to continue exploring.) Teacher uses student ideas and methods as the basis for lessons or mini-extensions.*
Students interject their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare, contrast, and build on ideas. Student ideas form part of the content of many math lessons.</td><td>*Teacher expects students to be responsible for co-evaluation of everyone's work and thinking, supports students as they help one another sort out misconceptions, and helps and/or follows up when needed.*
Students listen to understand, then initiate clarifying other students' work and ideas for themselves and for others during whole-class discussions as well as in small group and pair work. Students assist each other in understanding and correcting errors.</td></tr>
</table>

PRODUCTIVE DISPOSITION

Many aspects of the ***Math Expressions*** program support the crucial building of a productive disposition. The years of classroom research that underlie ***Math Expressions*** developed learning paths of supports and student strategies. These supports and strategies can move students from their initial knowledge to fluency with and understanding of formal mathematical methods and notation. ***Math Expressions*** works to facilitate the learning of general school competencies, to mobilize home-school links, and to build a helping community within the classroom. It sets high-level mathematical goals for all students and concentrates on prerequisite competencies to bring all children to mastery.

For all major grade-level topics, ***Math Expressions*** starts at the student's level and continually elicits their thinking, provides visual and linguistic supports to move them rapidly to understanding, and ends with extended fluency practice while continuing the emphasis on understanding and explaining. The curriculum is organized into ambitious core grade-level topics, cumulative experiencing, and peer helping throughout the year to bring students to a higher mathematical level. Meaning-focused classroom activities help students build knowledge and skills and move through developmental progressions to more efficient and general methods. Throughout, teachers use assessment results to adapt their teaching and to help students focus their learning.

Math Expressions also works to increase the self-regulatory actions of students, so they can become more organized (such as by doing work neatly and regularly completing homework) as well as set learning goals and carry out a plan to meet those goals. As they progress throughout the year, students increasingly come to recognize when they need to ask for help and to reflect on their own progress to affirm how much they have learned.

Mobilizing home-school links is essential to student success in any classroom, and it is an important part of the ***Math Expressions*** curriculum. Designating a "home helper" to monitor and help with daily homework is beneficial and easy—much of the homework is familiar to families and most is similar to that experienced by the student in the classroom. Teachers also provide home games and activities so students can learn and practice prerequisite competencies and knowledge skills.

Put together, all these teaching/learning methods culminate during daily Quick Practice activities in the classroom, which provide opportunity and structure for developing student leadership. Students work individually or in partners to build prerequisite skills and bring new skills to fluency. Eventually, all students take on leadership roles within the Quick Practice activities. Acting as a leader develops confidence in every student, regardless of ability level. Through these roles, students gradually assume more responsibility for the learning that takes place in the classroom.

A key aspect of this program is that everyone is both a teacher and a learner. Students learn how to be helpful members of a math community as they work and talk together. The classroom accepts each student's cultural world as part of this collaborative math learning.

National Science Foundation Research Results Reflected in *Math Expressions*

THE NATIONAL SCIENCE FOUNDATION

For more than fifty years, the National Science Foundation (NSF) has solicited and funded research on mathematics, science, engineering, and technology. Since the early 1980s NSF has led the way in encouraging mathematics curricula focused on inquiry and understanding by supporting the "cultivation of a research base for implementing innovative elementary mathematics…reform strategies," and has sought out "sound, innovative, promising, high risk/high payoff ideas that advance knowledge."

NSF receives approximately 40,000 applications per year. Researchers and educators submitting research proposals to the agency must address, in a significant way, several areas in order to be funded by the agency:

- Development of new knowledge, new tools, and innovative interpretations that enable significant impacts on policies and practices, thereby improving learning by students and whole systems
- Practices that grow the capacity of institutions to scale up and sustain change or best practices at all levels of education
- Professional development of the instructional workforce
- Contributions to the body of research already in existence on the topic

Research proposals are reviewed by a team of advisers, considered experts in their fields, from both inside and outside NSF. The advisory group, which focuses on program directions as well as specific proposals, involves approximately 50,000 advisers each year. Funding is dispersed over a period of time and well-defined milestones are required in order for support from NSF to continue.

THE NATIONAL SCIENCE FOUNDATION GRANTS THAT FUNDED THE DEVELOPMENT OF *MATH EXPRESSIONS*

The research behind the *Children's Math Worlds (CMW)* research project, from which the ***Math Expressions*** program was developed, was funded by two research grants and one materials development grant from NSF. The *CMW* research project is the only NSF elementary curriculum that was supported by both kinds of grants and that was progressively refined with at least four cycles of revision at each grade level. Research results on separate math topic areas were used immediately and directly within the developing program. They then were revised over years to adapt to a wide range of students, teachers, and schools, and to increasingly integrate across grades and math topic areas to create a highly coherent Kindergarten through Grade 5 program.

The three NSF grants from 1993 to 2003 were coordinated to fund a continuing program of research and materials development designed to enable all children, including those most at risk of mathematical failure, to learn mathematics that is substantially more advanced than is usual in this country. The first two years of research were primarily in urban English- and Spanish-language classrooms, but the final eight years expanded to include students from middle- and high-income families in suburban schools, students from many cultural backgrounds, and many native English speakers as well as students learning English. The *CMW* research identified developmental progressions of children's solution methods in many math topic areas that could be used as a basis for designing instructional materials. In collaboration with classroom teachers, the project developed a new teaching-learning approach called Mathematics Equity Pedagogy, which is used with student math drawings and Math Talk. This pedagogy permits children to learn a higher than ordinary level of mathematics by starting at students' current level and building ladders to mathematical concepts, symbols, and methods through students' use of language and meaningful math models that they draw and discuss. The goal was to develop materials that will work with the whole range of children in the United States, while ensuring success with those groups that have been under-represented in math and science occupations.

The teaching materials were developed from the extensive research base on student thinking, and the *CMW* project continually contributed to that research base. The program's frameworks facilitate student and teacher learning. The needs of the less-advanced and the more-advanced learners were balanced by working on ambitious mathematical topics in the grade levels at which they are taught in other countries, but the *CMW* research identified research-based ladders of support to help less-advanced learners progress. The development of the extensive use of student math drawings and Math Talk enables students at various levels to work on and discuss the same mathematical concepts at the same time.

Additional work developed balanced paths through single-digit and multi-digit addition, subtraction, multiplication, division, fraction, data, geometry, and measurement understandings for all students. An in-depth ambitious algebraic approach to word problems follows a learning path across the grades and continually is coordinated with understanding of and development of fluency with computation. Research-based strategies and accessible algorithms are taught so that all students can learn. Students also develop and explain their own strategies. The research-based accessible algorithms relate easily to common algorithms, which are also discussed. In all math topic areas, students make math drawings and engage in Math Talk to build understandings and linguistic competence and confidence. Students also engage in carefully designed progressions to build fluency. Daily Quick Practice activities are often conducted by student leaders, and students engage in various helping roles within the classroom community.

In all topic areas, the research focused on building a detailed map of the development of student strategies and errors, and how the project could support more advanced strategies and reduce errors. It was successful in doing this in major areas of the elementary math curriculum. Papers summarizing these results are listed in the final section of this document (p.21).

This brief summary includes results in two math topic areas: children's single-digit multiplication strategies and fraction concepts and computation.

For multiplication, the project discovered that student learning and conceptual development in multiplication is quite different from that in addition and subtraction. In addition and subtraction, student strategy development is driven mainly by general conceptual developments resulting from experience in adding and subtracting. A relatively invariant worldwide sequence of general strategies exists; this sequence reflects increasing internalization, abbreviation, and abstraction. However, this project's research on multiplication had quite different results. It indicated that multiplication strategy development is primarily driven by the acquisition of a large number of cognitive resources that are associated with specific numerical values. This then implies substantial school time to build this specific knowledge base with extra time for the most difficult numbers. These differing research results are reflected in the ***Math Expressions*** program. For addition and subtraction, students are helped to move through the worldwide sequence of solution strategies to those that are general and rapid enough to solve all single-digit addition and subtraction problems. For multiplication, students are helped to develop all the specific cognitive knowledge about particular numerical multiplications through an extensive set of coordinated conceptual and practice activities.

For fractions, research indicated that emphasizing the concept of unit fractions (1/D) as things in the world resulting from dividing into equal (D) shares helps students see fractions as quantities rather than as two numbers written with a bar between them. Adding, subtracting, and comparing fractions is then developed as adding, subtracting, and comparing combinations of unit fractions (e.g., 1/5 + 1/5 + 1/5 = 3/5). Equivalent fractions are developed as equivalence chains coming from two rows of the multiplication table, with any particular pairs of such equivalent fractions created by multiplying their ones column by their column number. Students can use the multiplication table to find equivalent fractions initially, enabling them to add and subtract fractions and mixed numbers with unlike denominators. This provides a conceptual basis for understanding that each fraction has many equivalent fractions generated by multiplying the top and bottom numbers in the fraction by the same number.

The work on multiplication and division of fractions enables students to discuss how these operations are like and different from the operations with whole numbers. Multiplying fractions involves multiplying numerators and denominators, but when the multiplying fraction is less than one, the initial number will decrease (rather than increase as with whole-number multiplication) because students are finding a part of it. Dividing fractions involves dividing numerators and denominators, and when the dividing fraction is less than one, the initial number will increase (rather than decrease as with whole-number division) because students are finding how many small parts are in it. This approach addresses major misconceptions about multiplication and division of fractions that have been found to be pervasive in students' work and to interfere very heavily in their later work in algebra.

All fraction work uses number bar and number line length models. The project found that many students have great difficulty learning to use number line models correctly. However, number lines are part of mathematical notation, so they are gatekeeper knowledge. We have found that sustained work with number lines across all fraction operations enables most students to build mastery of this notation. Students using these approaches have outscored U.S. students using traditional textbooks and have been equivalent to Chinese and Japanese students.

The project's research on proportion, as with all earlier topics researched, addressed the core misconceptions and errors found in earlier research. In this case, students add rather than multiply when solving proportions. So the curriculum begins by having students work with the multiplication table, finding patterns in it. They then look at a range of multiplication situations as rate situations, making rate tables. Ratio situations are then seen as two rate situations that are linked. Students make rate tables and ratio tables by using vertical columns cut from a multiplication table. This approach grounds proportion heavily in multiplication, and research finds that very few students make addition errors. MT puzzles, four corners of a rectangle from the multiplication table, are also identified. These actually form a proportion. Students solve many such MT puzzles in which one number is missing. This draws on and helps develop their basic multiplication and division facts. Students also fill in scrambled multiplication tables in which the rows and columns have been disordered. Students at all levels love these MT puzzles and scrambled multiplication tables and work on them enthusiastically. This massive practice, and feedback from it, results in considerable improvement in students' multiplication and division facts, as well as in their successful solving of proportion problems that are considerably more complex than those ordinarily solved by Grade 5 students.

When research began on ratio and proportion, proportion was one of the earliest hurdles to successful performance in middle school math. However, the research found that multiplication itself is a major hurdle. A surprising number of fourth and fifth graders from all backgrounds do not really understand multiplication or do not trust it enough to use it. They will add repeatedly even when they know the related multiplication fact. Others do not yet know their multiplication facts. This program's approach to proportion simultaneously enables less-advanced students to learn and feel comfortable with multiplication while also learning to solve proportion problems. More-advanced students learn multiple ways to solve rate and ratio situations.

This curriculum's approaches to multi-digit multiplication, multi-digit division, fractions, ratio, and proportions work with both less-advanced and more-advanced students because they are all approached as additive multiplication. It has outlined students' early conceptions of multiplication in four areas that are termed additive multiplication. Students can use their knowledge of addition to approach these domains initially, and then receive help in transitioning to a more advanced view involving multiplication perspectives. More-advanced students begin with, learn, and relate multiplicative solution methods in all of these topic areas.

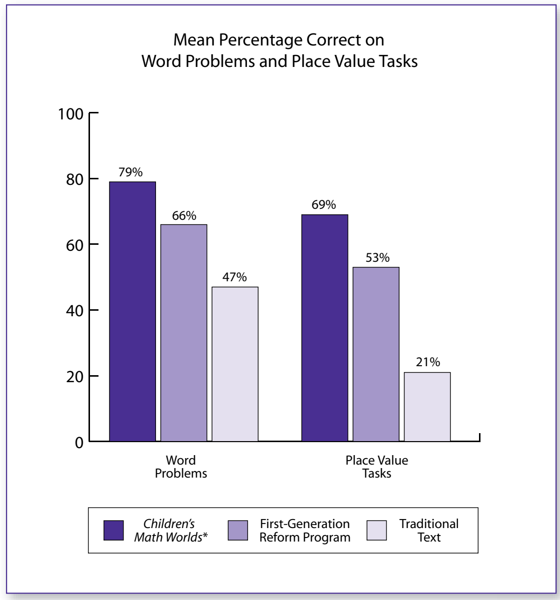
As with the topics for the earlier grades, research on more advanced topics in Grades 3, 4, and 5 developed and tested in urban and suburban classrooms bridges to mathematical concepts, symbols, and methods that used linguistic supports and meaningful drawn math models.

It found that the multiplication table itself is a helpful bridge, not only for the basic multiplications, but also for fractions and ratio. This use has high face validity with teachers and students, enabling a level of comfort with these challenging topics.

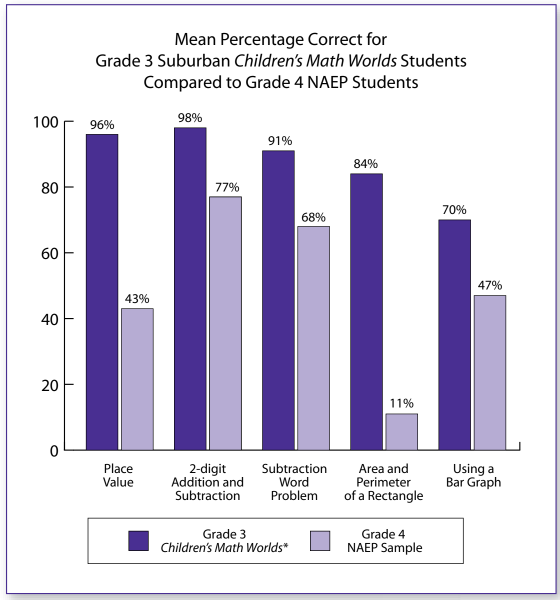
In all studies on particular core concept domains from Kindergarten through Grade 5, *CMW* students exceeded the performance of comparison groups by 10 percent to 40 percent and often were comparable to or exceeded the performance of students one to four years ahead. Specific research articles are organized by math topic area with summary comments in the last section of this document.

Over the research period, data was obtained on student performance on four different kinds of standardized tests. Performance of *CMW* students from Grades 1 through 5 exceeded that of students in the school who had earlier used traditional textbooks.

Houghton Mifflin is working in partnership with early adopters of the program to obtain preliminary evidence of its effectiveness in its current ***Math Expressions*** form. These early learner verification studies are part of a larger research agenda, which includes pretest and posttest comparisons, matched comparison studies, and experimental and quasi-experimental designs. These research results will add to the large research base on successful design approaches and student learning paths, in particular mathematical topic areas that are reported in the papers in the last section of this framework.



****Math Expressions*** is the curriculum developed from the *Children's Math Worlds Research Project.*



****Math Expressions*** is the curriculum developed from the *Children's Math Worlds Research Project.*

Evaluation of the Impact of ***Math Expressions*** on Student Performance

Building on the NSF-funded research that author Karen Fuson conducted during the development of ***Math Expressions,*** Houghton Mifflin is working in partnership with early adopters of the program to obtain additional evidence of its effectiveness. Houghton Mifflin has contracted with an independent research firm to oversee data collection and complete the analysis of test scores and teacher responses to the survey. Results from these studies will be included in an updated version of *A Research-Based Framework for* ***Math Expressions:*** *Grades K–5.*

Anecdotal Reports from Teachers Using *Math Expressions*

During the *Children's Math Worlds Research Project (CMW)*, the results of which were published in the curriculum called ***Math Expressions,*** all teacher participants improved their knowledge, teaching skills, and attitudes as measured by ongoing discussion in meetings, end-of-year questionnaires, observation of classroom teaching by project staff, and individual interviews. The improvements in knowledge were in understanding of mathematical content, children's mathematical thinking and language, and conceptualizations of teaching.

During the project, teachers in *CMW* classrooms moved toward reform teaching. They were asked to observe how their students conceptualized mathematical problems and to pursue issues of mathematical thinking. All teachers indicated that they had gained substantial knowledge about the various ways in which students solve problems. All said they had improved their teaching skills by leading discussions about students' mathematical thinking. All but one teacher showed substantial gains in classroom support of students linking everyday and mathematical language. All teachers began to engage their students in making math drawings (mathematical modeling) as part of the problem-solving process. Changes in attitudes included higher expectations of what their students could do mathematically (because they saw their students doing much more as a result of the curriculum), more enjoyment of teaching mathematics, greater relation of math to aspects of their students' lives, and more interest in mathematics and problem solving in their own lives.

Sample comments from teachers about their own learning with *CMW*:

"The approach in *CMW* helps orient me as to how to present/phrase concepts to kids in language that makes sense to them."

"I have a much deeper understanding of all the concepts in primary-level math."

"Since *CMW* emphasized process, I have become aware of my own ways of solving problems and can see the link in my own math life!"

"I learned that children's mathematical thinking expands more quickly and thoroughly in real-world situations."

"I enjoy it when my children come up to the board and express themselves. This gives me a clearer idea of their learning."

"I used to think that some word problems were too difficult for primary-grade students. Now that I've taught first-grade students to draw and label their solutions to word problems, I will be more careful to include many types of word problems (including comparisons and multi-step problems) in my math lessons."

"I love teaching math, most of all because it is fun and my students enjoy it so much."

"It was so interesting to see the many different ways that children approached math. The *CMW* materials allow for a variety of responses."

"I was amazed [by] the solution process—how explaining their methods really helped my students learn."

"I really felt more interested in teaching math and felt a connection between the concepts."

A sample of teacher comments about student learning and attitudes with *CMW*:

"I have many kids who cheer when it's math time! They all do the homework and generally are psyched."

"My students are strong conceptually and are getting stronger in language abilities and practice-type things."

"They understand the concept of tens and ones (place value) much better with *CMW*."

"I think the children really enjoyed and benefited from the active engagement."

"Children seemed to have very happy, positive attitudes about coming to math."

"The students' understanding of why we use certain procedures to solve equations was improved."

"Higher-order thinking skills were developed more than in previous years."

"My students are better able to speak and think about the solving process. This helps them understand and attack the problem with better understanding."

"My children ask me all day, 'When is it math time?' Or, 'Can I just go ahead and finish up the book myself?'"

"My students' problem-solving skills were better than in past years. They were able to understand the situation and the question much better. They also understood and were able to add multi-digit numbers with ease."

"Their confidence improved, as did their attention span and homework completion rate."

"Because students interact with one another, communication and positive relationships grew and more than one solution was found. This all came from the students themselves."

"When children are given hands-on experiences and are taught and presented concepts in a variety of ways, they [more easily] retain and understand those concepts."

"The mathematical growth was tremendous and the children had a great attitude about math. I would say their overall comment about math is, 'It is fun!'"

"The language experiences are crucial in mathematical development and these experiences are presented throughout all grade levels in this program."

"My students definitely enjoyed math a lot. They loved doing word problems and were confident about their procedures and answers."

Additional teacher comments about *CMW*:

"I've brought my less-advanced first graders to where I used to bring my top children."

"*CMW* continues to be the bright spot of the day for my second-grade class. We just love it!"

"The step-by-step breakdown and hands-on activities make it ideal for adaptation to special-education students."

"The approaches that *CMW* uses to teach math are truly very successful, and I have seen the confidence of students increase to the point that they are no longer afraid to experiment or try to figure problems out. I have seen the children learn to love math rather than fear it and this is also reflected in the reactions of the parents. Parents become active participants in their children's learning process and they are able to see how their children are progressing and becoming excited about math."

"Having Homework Helpers assigned from the beginning lets parents know the importance of what their child is learning in class, as well as the importance of the parents' involvement in that learning process (in and out of class)."

"*CMW* is a wonderful way to teach students how to cooperate, communicate, and calculate—all in a fun and interactive way."

"When I started using the *CMW* curriculum in my classroom, I observed students achieving greater levels of understanding in all areas of mathematics. I have watched students react much more positively to math and their feelings of success have increased substantially."

"Math is seen as relevant and all around us. It is no longer something that we do an hour a day in the classroom, but something that we see and use every day. Children are excited and look forward to doing math."

Project References and the Nature of These Papers

National Research Council. *Adding It Up: Helping Children Learn Mathematics*. J. Kilpatrick; J. Swafford; and B. Findell, eds. Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, D.C.: National Academy Press, 2001.

National Research Council. *Helping Children Learn Mathematics*. J. Kilpatrick and J. Swafford, eds. Center for Education, Division of Behavioral and Social Sciences and Education. Washington, D.C.: National Academy Press, 2002.

The following papers describe 1) research on teaching and learning that provides part of the research base from which the *Children's Math Worlds Research Project (CMW*) was developed and 2) research reports that document the individual design studies and their success with students. More research papers exist in draft and research summary form based on the intensive ten-year period of progressive refinement of the curriculum using extensive observations and feedback from teachers, and more papers will be written to summarize research results in other areas. The *CMW* program has been published as ***Math Expressions.***

RESEARCH CONCERNING BROAD ASPECTS OF THE *CMW* CURRICULUM

A.

Hufferd-Ackles, K.; K.C. Fuson; and M.G. Sherin. "Describing Levels and Components of a Math-Talk Community." *Journal for Research in Mathematics Education* 35(2):81–116.

Fuson, K.C.; Y. De La Cruz; S. Smith; A. Lo Cicero; K. Hudson; P. Ron; and R. Steeby. Blending the Best of the 20th Century to Achieve a Mathematics Equity Pedagogy in the 21st Century. In M.J. Burke and F.R. Curcio, eds. *Learning Mathematics for a New Century*. Reston, VA: National Council of Teachers of Mathematics, 2000, pp. 197–212.

The above two articles describe central aspects of ***Math Expressions*** classrooms in action to create communication, confidence, and competence.

B.

Fuson, K.C. Developing Mathematical Power in Whole Number Operations. In J. Kilpatrick; W.G. Martin; and D. Schifter, eds. *A Research Companion to Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics, 2003, pp. 68–94.

Fuson, K.C. Research on Whole Number Addition and Subtraction. In D. Grouws, ed. *Handbook of Research on Mathematics Teaching and Learning.* New York: Macmillan, 1992, pp. 243–275.

In both of these articles, research concerning effective ways to teach computation and word problems is summarized. ***Math Expressions*** (K–5) draws from this research base, using the full range of word problem types found in many countries.

C.

Fuson, K.C. Pre-K to Grade 2 Goals and Standards: Achieving Mastery for All. In D.H. Clements; J. Sarama; and A.M. DiBiase, eds. *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc., 2004, pp.105–148.

This chapter describes aspects of the design of the ***Math Expressions*** program and the reasons for this design. Deep core, ambitious grade-level topics are chosen as the center of the curriculum at each grade level, and other topics are woven around these, permitting high levels of coherence within and across grades. Early research related to this topic is in K.C. Fuson; J. Stigler; and K. Bartsch. "Grade Placement of Addition and Subtraction Topics in China, Japan, the Soviet Union, Taiwan, and the United States." *Journal for Research in Mathematics Education* 19:449–458.

D.

Professor Fuson served on the National Research Council's Math Learning Committee that produced the report on K–8 math learning, *Adding It Up*, and its related summary *Helping Children Learn Math*. The *CMW* curriculum is designed to develop mathematical proficiency as defined by these reports: it continually interweaves understanding, computing, applying, reasoning, and engaging. *CMW* uses accessible algorithms, as discussed in the report. These are computational algorithms that have been demonstrated to be comprehensible by students and whose discussion in class can contribute to deeper understanding of place value and of computational methods. It also supports learning paths that help students move from initial slower, more concrete solution methods to general, rapid-enough methods.

Professor Fuson also wrote the introductory chapter for the National Research Council's book for teachers based on its report *How People Learn*. This is available with the other mathematics chapters in paperback alone or in hardback with the chapters on history and science.

Fuson, K.C; M. Kalchman; and J.D. Bransford. Mathematical Understanding: An Introduction. In M.S. Donovan and J.D. Bransford, eds. *How Students Learn: Mathematics in the Classroom*. Washington, D.C.: National Academy Press, 2005, pp. 217–256. [This paperback book contains the introductory and final chapter and the four mathematical chapters from the next book.]

Fuson, K.C.; M. Kalchman; and J.D. Bransford. Mathematical Understanding: An Introduction. In M.S. Donovan and J.D. Bransford, eds. *How Students Learn: History, Math, and Science in the Classroom*. Washington, D.C.: National Academy Press, 2005, pp. 217–256.

E.

Fuson, K.C.; A. Lo Cicero; K. Hudson; and S.T. Smith. Snapshots Across Two Years in the Life of an Urban Latino Classroom. In J. Hiebert; T. Carpenter; E. Fennema; K.C. Fuson; D. Wearne; H. Murray; A. Olivier; and P. Human. *Making Sense: Teaching and Learning Mathematics with Understanding*. Portsmouth, NH: Heinemann, 1997, pp. 129–159.

Lo Cicero, A.; K.C. Fuson; and M. Allexaht-Snider. Making a Difference in Latino Children's Math Learning: Listening to Children, Mathematizing Their Stories, and Supporting Parents to Help Children. In L. Ortiz-Franco; N.G. Hernandez; and Y. De La Cruz, eds. *Changing the Faces of Mathematics: Perspectives on Latinos*. Reston, Virginia: National Council of Teachers of Mathematics, 1999, pp. 59–70.

These articles describe central aspects of developing a ***Math Expressions*** classroom. The first NSF grant research work was primarily in urban Latino classrooms, but the later two NSF grants developed and tested the curricular approaches in a wide range of schools including advantaged schools with many high-achieving students.

RESEARCH CONCERNING THE APPROACHES TO SPECIFIC TOPICS IN *CMW*

Learning in Kindergarten

Ho, C.S., and K.C. Fuson. "Effects of Language Characteristics on Children's Knowledge of Teens Quantities as Tens and Ones: Comparisons of Chinese, British, and American Kindergartners." *Journal of Educational Psychology* 90:536–544.

Fuson, K.C.; L. Grandau; and P. Sugiyama. "Achievable Numerical Understandings for All Young Children." *Teaching Children Mathematics* 7(9):522–526. Invited paper for the "Research Into Practice" series.

These papers report aspects of the design of the ***Math Expressions*** Kindergarten program so that it can support ambitious international levels of conceptions of teen numbers as tens and ones as well as embedded numbers that can support advanced addition and subtraction methods.

Powerful and general finger and mental methods for adding and subtracting numbers to 20

Fuson, K.C.; T. Perry; and Y. Kwon. Latino, Anglo, and Korean Children's Finger Addition Methods. In J.E.H. van Luit, ed. *Research on Learning and Instruction of Mathematics in Kindergarten and Primary School*. Doetinchem/Rapallo: Graviant, 1994, pp. 220–228.

Fuson, K.C., and W.G. Secada. "Teaching Children to Add by Counting On with Finger Patterns." *Cognition and Instruction* 3:229–260.

Fuson, K.C. "Teaching Children to Subtract by Counting Up." *Journal for Research in Mathematics Education* 17:172–189. This paper was chosen as the best research article of 1986 by the Research Advisory Council of the National Council of Teachers of Mathematics.

Fuson, K.C. "Adding by Counting On with Finger Patterns." *Arithmetic Teacher* 35:38–41. Invited paper; first article in the new "Research Into Practice" series of articles.

Fuson, K.C. Subtracting by Counting Up with Finger Patterns. *Arithmetic Teacher* 35(5):29–31. Invited paper for the "Research Into Practice" series.

Fuson, K.C., and Y. Kwon. "Korean Children's Single-Digit Addition and Subtraction: Numbers Structured by Ten." *Journal for Research in Mathematics Education* 23:148–165.

Duncan, A; H. Lee; and K.C. Fuson. Pathways to Early Number Concepts: Use of 5- and 10-Structured Representations in Japan, Taiwan, and the United States. In M.L. Fernandez, ed. *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2000, p. 452.

Murata, A., and K.C. Fuson (2001). Learning Paths to 5- and 10-Structured Understanding of Quantity: Addition and Subtraction Solution Strategies of Japanese Children. In R. Speiser; C.S. Maher; and C. Walter, eds. *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2001, pp. 639–646.

Murata, A., and K.C. Fuson. "Teaching as Assisting Individual Constructive Paths Within an Interdependent Class Learning Zone: Japanese First Graders Learning to Add Using Ten." *Journal for Research in Mathematics Education* (in press).

This and related research on different ways in which children around the world use fingers for a developmental sequence of solution strategies is woven into the ***Math Expressions*** program to support children's natural ways of thinking while moving them on to effective, rapid, and mathematically general methods.

Place value concepts and multi-digit addition and subtraction methods

Fuson, K.C., and Y. Kwon. Chinese-Based Regular and European Irregular Systems of Number Words: The Disadvantages for English-Speaking Children. In K. Durkin and B. Shire, eds. *Language and Mathematical Education*. Milton Keynes, GB: Open University Press, 1991, pp. 211–226.

Fuson, K.C. “Issues in Place-Value and Multidigit Addition and Subtraction Learning.” *Journal for Research in Mathematics Education* 21:273–280.

Fuson, K.C. “Conceptual Structures for Multiunit Numbers: Implications for Learning and Teaching Multidigit Addition, Subtraction, and Place Value.” *Cognition and Instruction* 7:343–403.

Fuson, K.C., and Y. Kwon. “Korean Children’s Understanding of Multidigit Addition and Subtraction.” *Child Development* 63:491–506.

Fuson, K.C., and S.T. Smith. “Complexities in Learning Two-Digit Subtraction: A Case Study of Tutored Learning.” *Mathematical Cognition* 1:165–213.

Fuson, K.C., and S.T. Smith. Supporting Multiple 2-Digit Conceptual Structures and Calculation Methods in the Classroom: Issues of Conceptual Supports, Instructional Design, and Language. In M. Beishuizen; K.P.E. Gravemeijer; and E.C.D.M. van Lieshout, eds. *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures.* Utrecht, The Netherlands: CD-B Press/The Freudenthal Institute, 1997, pp. 163–198.

Fuson, K.C.; D. Wearne; J. Hiebert; P. Human; H. Murray; A. Olivier; T. Carpenter; and E. Fennema. “Children’s Conceptual Structures for Multidigit Numbers at Work in Addition and Subtraction.” *Journal for Research in Mathematics Education* 28:130–162.

Fuson, K.C.; S.T. Smith; and A. Lo Cicero. “Supporting Latino First Graders’ Ten-Structured Thinking in Urban Classrooms.” *Journal for Research in Mathematics Education* 28:738–766.

Fuson, K.C., and A.M. Lo Cicero. El Mercado in Latino Primary Math Classrooms. In M.L. Fernandez, ed. *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2000, p. 453.

Fuson, K.C., and B.H. Burghardt. Multi-Digit Addition and Subtraction Methods Invented in Small Groups and Teacher Support of Problem Solving and Reflection. In A. Baroody and A. Dowker, eds. *The Development of Arithmetic Concepts and Skills: Constructing Adaptive Expertise*. Hillsdale, NJ: Erlbaum, 2003, pp. 267–304.

These research studies and related research describe issues and approaches that underlie the ***Math Expressions*** approach to leading students to understanding and fluency with place value concepts and multi-digit addition and subtraction algorithms.

Teaching word problems

Stigler, J.; K.C. Fuson; M. Ham; and M.S. Kim. "An Analysis of Addition and Subtraction Word Problems in Soviet and American Elementary Textbooks." *Cognition and Instruction* 3:153–171.

Fuson, K.C. First and Second Graders' Ability to Use Schematic Drawings in Solving Twelve Kinds of Addition and Subtraction Word Problems. In M.J. Behr; C.B. Lacampagne; and M.M. Wheeler, eds. *Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. DeKalb, IL: Northern Illinois University, 1988, pp. 364–370.

Fuson, K.C., and G.B. Willis. "Second Graders' Use of Schematic Drawings in Solving Addition and Subtraction Word Problems." *Journal of Educational Psychology* 81:514–520.

Fuson, K.C.; W.M. Carroll; and J. Landis. "Levels in Conceptualizing and Solving Addition/Subtraction Compare Word Problems." *Cognition and Instruction* 14(3):345–371.

Lo Cicero, A; Y. De La Cruz; K.C. Fuson. "Teaching and Learning Creatively with the *Children's Math Worlds* Curriculum: Using Children's Narratives and Explanations to Co-Create Understandings." *Teaching Children Mathematics* 5(9):544–547.

These research studies and related research describe issues and approaches that underlie the ***Math Expressions*** approach to assisting students in solving ambitious international levels of word problems at all grade levels. The detailed Kindergarten through Grade 5 learning path for word problems in ***Math Expressions*** will be described in a forthcoming paper.

Multiplication and Division

Sherin, B., and K.C. Fuson. "Multiplication Strategies and the Appropriation of Computational Resources." *Journal for Research in Mathematics Education* 36(4):347–395.

This outlines student strategies for learning single-digit multiplication and division. The *CMW* Grades 3, 4, and 5 have work in these areas based on this research.

Izsák, A., and K.C. Fuson. Students' Understanding and Use of Multiple Representations While Learning Two-Digit Multiplication. In M.L. Fernandez, ed. *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2000, pp. 714–721.

Izsák, A. Learning Multi-Digit Multiplication by Modeling Rectangles. In R. Speiser; C. Maher; and C. Walter, eds. *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2001, pp. 187–194.

Izsák A. "Teaching and Learning Two-Digit Multiplication: Coordinating Analyses of Classroom Practices and Individual Student Learning." *Mathematical Thinking and Learning* 6(1):37–79.

Izsák, A. "'You Have to Count the Squares': Applying Knowledge in Pieces to Learning Rectangular Area." *The Journal of the Learning Sciences* 14(3):361–403.

Izsák, A., and Sherin, M. "Exploring the Use of New Representations as a Resource for Teacher Learning." *School Science and Mathematics* 103(1):18–27.

Sherin, M., and Izsák, A. Representations as a Resource for Teacher Learning. In R. Speiser; C. Maher; and C. Walter, eds. *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2001, pp. 961–962.

These research studies and related research describe issues and approaches that underlie the ***Math Expressions*** approach to assisting students with initial single-digit multiplication and division concepts and fluency place value concepts, and then to understanding and fluency with multi-digit multiplication and division algorithms.

Fractions, Ratio, and Proportion

Fuson, K.C., and M. Kalchman. A Length Model of Fractions Puts Multiplication of Fractions in the Learning Zone of Fifth Graders. In D.L. Haury, ed. *Proceedings of the Twenty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 2002, pp. 1641–1649.

Fuson, K.C., and D. Abrahamson. Understanding Ratio and Proportion as an Example of the Apprehending Zone and Conceptual-Phase Problem-Solving Models. In J. Campbell, ed. *Handbook of Mathematical Cognition*. New York: Psychology Press, 2005, pp. 213–234.

These research studies and related research describe issues and approaches that underlie the ***Math Expressions*** approach to leading students to understanding and fluency with fraction concepts and calculation, and with ambitious but accessible aspects of ratio and proportion.

Notes